

# Critical behavior of a three-state Potts model on a Voronoi lattice

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**Abstract.** We use the single-histogram technique to study the critical behavior of the three-state Potts model on a (random) Voronoi-Delaunay lattice with size ranging from 250 to 8000 sites. We consider the effect of an exponential decay of the interactions with the distance,  $J(r) = J_0 \exp(-ar)$ , with  $a > 0$ , and observe that this system seems to have critical exponents  $\gamma$  and  $\nu$  which are different from the respective exponents of the three-state Potts model on a regular square lattice. However, the ratio  $\gamma/\nu$  remains essentially the same. We find numerical evidences (although not conclusive, due to the small range of system size) that the specific heat on this random system behaves as a power-law for  $a = 0$  and as a logarithmic divergence for  $a = 0.5$  and  $a = 1.0$ .

**PACS.** 64.60.Fr Equilibrium properties near critical points, critical exponents – 05.10.Ln Monte Carlo methods – 05.20.-y Classical statistical mechanics

## 1 Introduction

The randomness in the lattice of statistical spin models has been studied in order to access the effect of impurities and dilutions over their critical behavior. It was conjectured by Harris [1] that the sign of the critical exponent of the specific heat,  $\alpha$ , determines whether the system is affected or not by such a randomness. For positive values of  $\alpha$ , the impure system should have a critical behavior different from the one of the pure system. For negative values of  $\alpha$ , on the other hand, the critical behavior of the system should be the same for both cases. In the marginal case of  $\alpha = 0$ , one should not be able to draw any conclusion about changes in the system.

The pure ferromagnetic three-state Potts model has  $\alpha = 1/3$ , hence, from the Harris criterium we expect to find a different behavior with a random interaction system. However, Picco [5] used this model with two types of interactions ( $J_0 = 1$  and  $J_1 = 1/10$ ) randomly and equally distributed ( $p = 0.5$ ) and did not find significant differences from the pure case. In another study [6], a quenched random interaction between nearest neighbors has been introduced, where the coupling factor  $J_{ij}$  for each pair  $i, j$  is selected from two positive values  $J$  and  $J'$  with respective probabilities equal to  $p$  and  $1 - p$ . It was found that the exponent  $\eta (= 2 - \gamma/\nu)$  does not depend on the disorder length, while  $\nu$  and  $\gamma$  vary continuously with this type of disorder, satisfying the concept of weak universality proposed by Suzuki [7].

In the present work, we investigate the effect of quenched disorder on the three-state Potts model using a two dimensional Voronoi-Delaunay network. This kind of disordered lattice exhibits a random coordination number that varies from 3 to  $\infty$ , depending on the number of sites. In addition, the distance  $r$  between nearest neighbors changes randomly from site to site. This geometrical feature is incorporated in our model by using a coupling parameter that is an exponential function of  $r$ ,  $J(r) \propto \exp(-ar)$ , with  $a > 0$ . In a previous work, we observed that the two-dimensional ferromagnetic Ising model with the same coupling mechanism and the same lattice displays critical behaviors for  $\alpha$ ,  $\beta$  and  $\nu$  that are independent of the value of  $a$  and are the same as those of the pure system [8]. Here we extend this analysis to the three-state ferromagnetic Potts model.

## 2 Model and simulation

The Voronoi construction or tessellation for a given set of points is defined as follows.  $N$  points are randomly placed in a square with side lengths equal to  $N^{1/2}$ . Then, for each point we determine the polygonal cell consisting of the region closer to this point than to any other. The points whose cells share an edge are considered neighbors. The dual lattice (Delaunay's) is composed by the vertices of these polygons. The lattice obtained by linking the neighbor sites is the Voronoi network.

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The Hamiltonian for the three-state ferromagnetic Potts model is given by

$$-KH = \sum_{\langle i,j \rangle} J_{ij} \delta(\sigma_i - \sigma_j),$$

where  $K = 1/k_B T$ ,  $T$  is the temperature,  $k_B$  is the Boltzmann constant and  $\delta$  is the Dirac's delta. The summation is performed over all pair of neighbors in the Voronoi's network and the spins can take values in the set  $\{1, 2, 3\}$ . We assume that

$$J_{ij} = J(r_i - r_j) = J_0 \exp(-a|r_i - r_j|),$$

where  $J_0$  is a dimensionless and positive interaction factor,  $r_i$  and  $r_j$  are the position vectors of the sites  $i$  and  $j$ , respectively, and  $a \geq 0$  is a model parameter.

For each grid size, we use the single-cluster algorithm [10] to simulate the system in the vicinity of the phase transition, *i.e.*, with  $K = K_{C_{\max}}$ , to determine the system finite size behavior.

The value of  $K_{C_{\max}}$  is determined in the following way. We find the first estimate  $K_0$  as the point of maximum specific heat obtained in the curve (specific heat against  $K$ ) constructed by simulations with  $K$  varying. The value of  $K_0$  is used to perform from  $R = 10$  to 20 independent simulations with  $10^6$  MC steps each, from which, using the single histogram algorithm [11], we construct a series of curves of  $C$  against  $K$ . We then average over all of these curves in order to find the specific heat curve as

$$C(K) = (1/R) \sum_{i=1}^R C^{(i)}(K), \quad (1)$$

where  $C^{(i)}(K) = K^2 N (\langle e^2 \rangle_K - \langle e \rangle_K^2)$  is computed using the single-histogram with the results of simulation  $i$ . Finally, we obtain, from the average curve, the maximum value of the specific heat,  $C_{\max}$ , and its abscissa  $K_{C_{\max}}$ .

In order to estimate the ratios  $\beta/\nu$  and  $\gamma/\nu$ , we use the fact that the magnetization at the inflection point and the susceptibility should scale, respectively, as

$$|\langle M \rangle|_{\text{inf}} = L^{-\beta/\nu} f(tL^{1/\nu}) \propto L^{-\beta/\nu}, \quad (2)$$

$$\chi_{\max}(L) = \chi(K_{C_{\max}}(L), L) = AL^{\gamma/\nu}, \quad (3)$$

and that

$$\left| \frac{d}{dK} \langle |M| \rangle \right|_{\text{max}} = L^{-\beta/\nu+1/\nu} f'(tL^{1/\nu}) \propto L^{(1-\beta)/\nu}. \quad (4)$$

Hence, the logarithmic derivatives should scale as

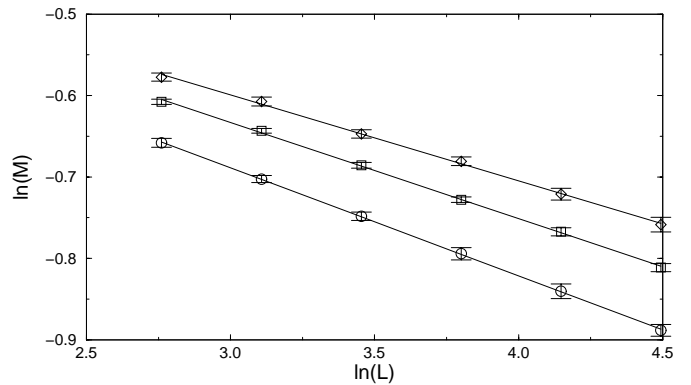
$$\left| \frac{d}{dK} \ln \langle |M| \rangle \right|_{\text{max}} = \left| \frac{1}{|M|} \frac{d}{dK} \langle |M| \rangle \right|_{\text{max}} \propto L^{1/\nu}, \quad (5)$$

and

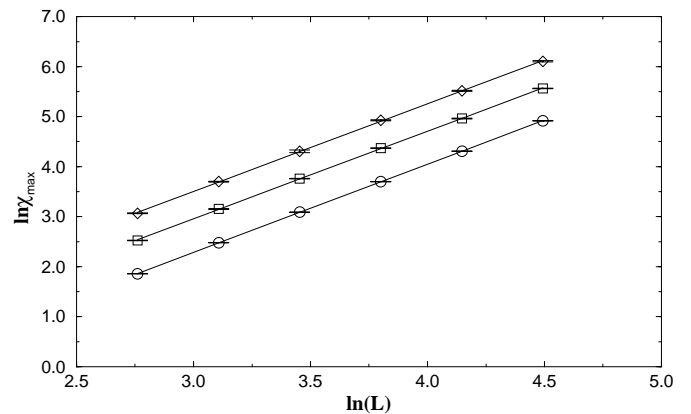
$$\left| \frac{d}{dK} \ln \langle |M^2| \rangle \right|_{\text{max}} = \left| \frac{1}{|M^2|} \frac{d}{dK} \langle |M^2| \rangle \right|_{\text{max}} \propto L^{1/\nu}, \quad (6)$$

and so we can get a good estimate of  $\nu$  from their scaling behavior. Finally, from finite size scaling arguments we can also predict that the specific heat should scale as

$$C_{\max}(L) = B_0 + B_1 \ln L. \quad (7)$$



**Fig. 1.** Logarithmic plots of magnetization ( $\ln(M)$ ) versus  $\ln(L)$  for  $a = 0.0$  (circles),  $0.5$  (squares) and  $1.0$  (diamonds).



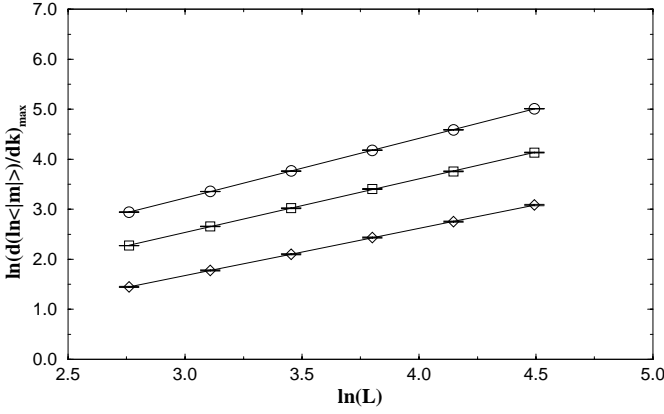
**Fig. 2.** Logarithmic plots of susceptibility ( $\ln(\chi)$ ) versus  $\ln(L)$  for  $a = 0.0$  (circles),  $0.5$  (squares) and  $1.0$  (diamonds).

### 3 Results and conclusions

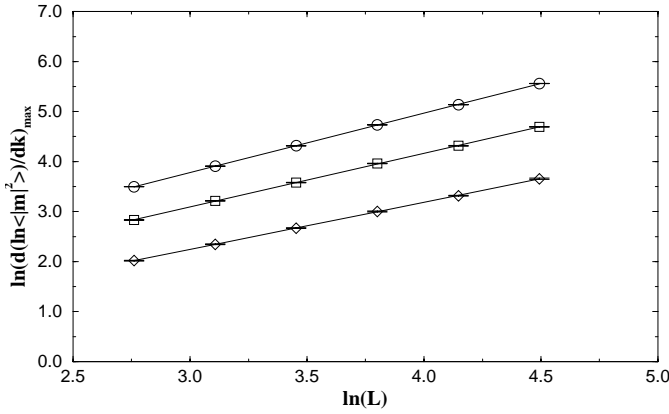
We study the critical behavior of the Potts model for three values of  $a$  ( $a = 0.0, 0.5$  and  $1.0$ ). For each value of  $a$ , we apply the finite size scaling technique [9] together with the single-histogram algorithm. We perform the same procedure for systems with different number of sites  $N = 250, 500, 1000, 2000, 4000$ , and  $8000$ . The critical temperature for infinite size system is estimated by using the fourth-order magnetization (Binder) cumulant and we find the critical values  $K_C = 0.607, 1.035$  and  $1.959$  and  $U^* = 0.606, 0.615$  and  $0.623$ , corresponding to  $a = 0.0, 0.5$  and  $1.0$ , respectively.

In Figure 1 we show a log-log plot of  $M$  against  $L (= N^{1/2})$  for  $a = 0.0, 0.5$  and  $1.0$ . By linear fitting each of these plots and using equation (2), we obtain  $\beta/\nu = -0.133, -0.118$  and  $-0.106$ , respectively. The errors in these measurements are in order of ten percent, which is true also for the plots of the other figures. This is a consequence of the small range of the system size.

In Figure 2 we show the plot of  $\chi_M$  against  $L$  also in logarithmic scale, from which we obtain the values of the exponent  $\gamma/\nu = 1.764, 1.751$  and  $1.754$  for  $a = 0.0, 0.5$



**Fig. 3.** Plots of  $\ln\left(\left|\frac{d}{dk}\ln\langle|M|\rangle\right|_{\max}\right)$  versus  $\ln(L)$  for  $a = 0.0$  (circles),  $0.5$  (squares) and  $1.0$  (diamonds).

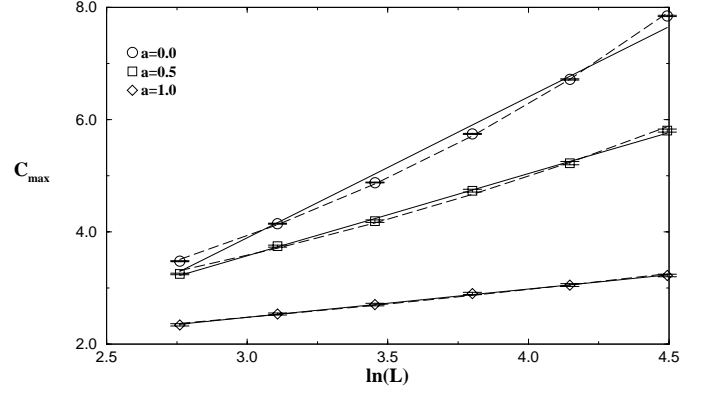


**Fig. 4.** Plots of  $\ln\left(\left|\frac{d}{dk}\ln\langle|M^2|\rangle\right|_{\max}\right)$  versus  $\ln(L)$  for  $a = 0.0$  (circles),  $0.5$  (squares) and  $1.0$  (diamonds).

and  $1.0$ , respectively. As in [6], we observe that both  $\gamma$  and  $\nu$  vary with  $a$  while the ratio  $\gamma/\nu$  remains essentially the same.

Figures 3 and 4 show the plots of the logarithmic derivatives (5) and (6) against  $\ln L$ . The slopes of the curves produce the estimates for  $1/\nu$ , from which we get the values  $\nu_1 = 0.840$ ,  $0.934$  and  $1.060$  and  $\nu_2 = 0.841$ ,  $0.934$  and  $1.061$  for  $a = 0.0$ ,  $0.5$  and  $1.0$  respectively.

In Figure 5 we show the plots of  $C_{\max}$  versus  $\ln L$  for values of  $a = 0.0$ ,  $0.5$  and  $1.0$ . We observe that the curves for  $a = 0.5$  and  $a = 1.0$  can be well fitted by a straight line while the curve for  $a = 0.0$  can not. The least-squares fits to data give the estimates  $B_0 = -0.818$  and  $0.956$  and  $B_1 = 1.464$  and  $0.506$  for  $a = 0.5$  and  $1.0$  respectively. Figure 5 also contains the exponential fittings of  $C_{\max}$  versus  $\ln L$ . From this figure, we can conjecture that, in the case of  $a = 0.0$ , the specific heat behaves like a power-law of  $L$ . In order to provide a quantitative support for this argument, we present in Table 1 the sum of the square errors obtained by the linear regressions (first row) and by the exponential fittings (second row) of  $C_{\max}\ln L$ .



**Fig. 5.** Plots of  $C_{\max}$  versus  $\ln(L)$  for  $a = 0$  (circles),  $0.5$  (squares) and  $1.0$  (diamonds). Also shown are the linear (solid lines) and exponential (dashed-lines) fittings to the data.

**Table 1.** Total square residuals of linear and exponential regressions of  $C_{\max}\ln L$  data.

$a$	0.0	0.5	1.0
Linear	1.262949e-01	6.017198e-03	7.392830e-04
Exponential	6.369170e-03	1.474494e-02	3.294589e-03

**Table 2.** Theoretical and computed values of the critical exponents.

	$\beta$	$\gamma$	$\nu$	$\beta/\nu$	$\gamma/\nu$
2d-Potts ( $q = 3$ )	1/9	13/9	5/6	2/15	26/15
$a = 0.0$	0.112	1.482	0.840	0.133	1.764
$a = 0.5$	0.110	1.635	0.934	0.118	1.751
$a = 1.0$	0.112	1.859	1.060	0.106	1.754

Table 2 contains our numerical results in a condensed form. From the analysis of this data, we conclude that the randomness, which is introduced here through the geometry of the Voronoi-Delaunay lattice, changes the critical behavior of the system. This can be clearly observed by the change in the values of  $\gamma$  and  $\nu$ . The value of  $\beta$  is not affected by the randomness. The analysis of the behavior of the exponent  $\alpha$  is not conclusive since it is difficult, due to the limited number of data points, to distinguish the trend (power-law or logarithmic) in the curve of  $C_{\max}$  versus  $L$ . As already mentioned, the fittings shown in Figure 5 provide some indication that the specific heat is a power-law of  $L$  for  $a = 0.0$ , and varies as a linear function of  $\ln L$  for  $a = 0.5$  and  $1.0$ . This conjecture needs to be checked, however, for values of  $L > 8000$ .

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